Decomposing numerical ranges along with spectral sets

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Abstract. This note is to indicate the new sphere of applicability of the method developed by Mlak as well as by the author. Restoring those ideas is summoned by current developments concerning K-spectral sets on numerical ranges.

The decomposition of numerical ranges the title refers to is, see [**13**, p. 42],

$$\mathfrak{W}(A \oplus B) = \operatorname{conv}(\mathfrak{W}(A) \cup \mathfrak{W}(B)); \tag{1}$$

it can be proved for any two Hilbert space operators A and B. The other decomposition is that of the spectrum of a function algebra related to a Hilbert space operator. These are the two leading topics of the current paper.

The algebra $\mathcal{R}(X)$

Here X stands always for a compact subset of $\mathbb{C}$ and $\mathcal{C}(X)$ does for the algebra of all continuous functions on X, with the supremum norm. Denote by $\mathcal{R}(X)$ the closure in $\mathcal{C}(X)$ of the algebra of all rational functions with poles off X. The *spectrum*[1] $\mathcal{X}$ of $\mathcal{R}(X)$, that is the set of all the characters (=nontrivial multiplicative functionals) of $\mathcal{R}(X)$, can be identified with X itself via the evaluation functionals (which are apparently characters)

$$X \ni x \mapsto \chi_x \in \mathcal{X}_{\mathcal{A}}, \quad \chi_x(u) \stackrel{\mathsf{def}}{=} u(x),\ u \in \mathcal{R}(X); \tag{2}$$

this mapping is in fact a homeomorphism with the topologies this of $\mathbb{C}$ and that of the $*$–weak topology of $\mathcal{R}(X)'$, the topological dual (see [**19**, Proposition 6.28] for a direct argument[2]).

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[1] Pretty often it is know under the name of the maximal ideal space of the algebra.

[2] From the monographs on function algebras, we recommend [**9**] as the most accessible one, also for missing here terminology; the source [**19**] has also to be mentioned.

A positive measure ν_x on X is called a representing measure of x (which is the same, according to the above identification, as the corresponding characters χ_x)

$$\chi_x(u) = u(x) = \int_X u \,\mathrm{d}\nu_x, \quad u \in \mathcal{R}(X).$$

The point mass at x is apparently a representing measure of x; if it is the only representing measure of x then x is called a *Choquet point* of $\mathcal{R}(X)$. All the Choquet points form the *Choquet boundary* Σ of $\mathcal{R}(X)$ and its closure is just the *Shilov boundary* Γ of $\mathcal{R}(X)$.

Sets of antisymmetry. Call subset A of X a *set of antisymmetry* of a function algebra $\mathcal{A}$ if, for u in $\mathcal{A}$, u real valued on A implies u is constant on A. Let $u|_A$ be the restriction of u to A, $\mathcal{A}|_A \stackrel{\text{def}}{=} \{u|_A\colon\ u \in \mathcal{A}\}$. Then we can restate Bishop's theorem [**4**] as

Theorem 1. *Every set of antisymmetry of $\mathcal{A}$ is contained in a maximal set of antisymmetry. The collection $\mathfrak{A}$ of maximal sets of antisymmetry forms a pairwise disjoint, closed covering of X satisfying*
(a) *$u \in \mathcal{C}(X)$ and $u|_A \in \mathcal{A}|_A$ for every $A \in \mathfrak{A}$ imply $u \in \mathcal{A}$;*
(b) *$\mathcal{A}|_A$ is closed in $\mathcal{C}(A)$, $A \in \mathfrak{A}$.*

An addition information we are going to use here comes form [**11**] (see also [**9**, Chapter II, Theorem 12.7] for more explicit exposition).

Theorem 2. *Every maximal set of antisymmetry A is an intersection of peak sets. A is an intersection of peak set if and only if $\mu \perp \mathcal{A}$ implies $\mu|_A \perp \mathcal{A}$.*

Gleason parts. Due to the identification established by (2) X considered as the spectrum of $\mathcal{R}(X)$ decomposes disjointly into the sets, called the *Gleason parts* of $\mathcal{R}(X)$, according to the relation[3] $\|\chi_{x_1} - \chi_{x_2}\| < 2$, $x_1, x_2 \in X$ with the norm being that of $\mathcal{R}(X)'$; this relation turns out to be an equivalence, cf. [**10**]. The crucial point is that if x_1 and x_2 are in the same Gleason part then there are measures ν_1 and ν_2 on Γ representing these points and such that $c\nu_2 \leqslant \nu_1$ and $c\nu_1 \leqslant \nu_2$ with some $0 < c < 1$, see [**5**]. This allows us to think of complex measures absolutely continuous with respect to a Gleason part G, in short *G–continuous*.

Denote by μ_G a G–continuous part of μ with respect to a Gleason part G. Let $(G_\alpha)_\alpha$ be the collection of all Gleason parts of $\mathcal{R}(X)$. It is known [**12**] (see also [**9**, ChapterVI, Theorem 3.4]) that

Theorem 3. *Suppose $X \subset \mathbb{C}$ is compact. Then*
(i) *for every $\mu \in \mathcal{M}(X)$ there exists a unique $\mu_0 \in \mathcal{M}(X)$ and a sequence (α_n) such that*

$$\mu = \sum\nolimits_n \mu_{G\alpha_n} + \mu_0$$

the sum being norm convergent;
(ii) *μ_{G_α} and μ_0 are in $\mathcal{A}^\perp$ provided so is μ;*
(iii) *$\mu_{G_\alpha} \perp \mu_{G_\beta} = 0$ if $\alpha \neq \beta$ and $\mu_{G_\alpha} \perp \mu_{G_0} = 0$ for all α.*

What relates Gleason parts to pick points is (see [**9**, ChapterVI, Theorem 3.1]).

Theorem 4. *If $\{x\}$, $x \in X$, is a peak set for $\mathcal{R}(X)$ then the Gleason part which contains x has a positive planar Lebesgue measure.*

[3] A full description of this relation, in the general case of a function algebra, can be found for instance in [**19**, Theorem 3.12], see [**4**] and [**2**] for the master results.

Representations of $\mathcal{R}(X)$ and their elementary spectral measure

An algebra homomorphism Φ of $\mathcal{R}(X)$ into $\boldsymbol{B}(\mathcal{H})$, the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$, is called a *representation of* $\mathcal{R}(X)$ *on* $\mathcal{H}$ if it is bounded and $\Phi(1) = I$. It follows from the Hahn–Banach theorem and Riesz representation theorem that for every $f, g \in \mathcal{H}$ there a complex measure $\mu_{f,g}$ such that

$$\langle \Phi(u)f, g\rangle = \int_X u \,\mathrm{d}\mu_{f,g}, \quad u \in \mathcal{R}(X),$$
$$\|\mu_{f,g}\| \leqslant \|\Phi\|\|f\|\|g\|.$$

Call any system $\{\mu_{f,g}\}_{f,g\in\mathcal{H}}$ that of *elementary spectral measure of* Φ; we refer also to the system $\{\mu_{f,g}\}_{f,g\in\mathcal{H}}$ as elementary measures of the operator

$$T \overset{\mathsf{def}}{=} \Phi(u_1) \text{ where } u_1(z) = z \text{ on } X. \tag{3}$$

With notation $u_1(z) = z$ one can show immediately that the spectrum $\mathrm{sp}(T)$ of $T \overset{\mathsf{def}}{=} \Phi(u_1)$ is contained in X.

Given a bounded projection (=idempotent) Q on $\mathcal{M}(X)$, the dual of $\mathcal{C}(X)$. We say that Q has the *property R*, after F. and M. Riesz, if

$$\mu \in \mathcal{R}(X)^{\perp} \implies Q\mu \in \mathcal{R}(X)^{\perp}$$
$$uQ\mu = Q(u\mu), \quad u \in \mathcal{C}(X),\ \mu \in \mathcal{M}(X). \tag{4}$$

The important observation, see [**22**], p. 102 and more in Section 3 of [**20**], is that for a system $\boldsymbol{Q}$ of commuting projections having the property R so does any member of the Boolean algebra $\mathcal{B}(\boldsymbol{Q})$ of projections the system generates.

For a system $\{\mu_{f,g}\}_{f,g}$ of elementary spectral measures of a representation Φ one may try to define a representation Φ_Q by

$$\langle \Phi_Q(u)f, g\rangle \overset{\mathsf{def}}{=} \int_X u \,\mathrm{d}Q\mu_{f,g}, \quad u \in \mathcal{R}(X),\ f, g \in \mathcal{H}.$$

If the projection Q has the property R, then Φ_Q is uniquely determined and $P_Q \overset{\mathsf{def}}{=} \Phi_Q(1)$ is a projection in $\boldsymbol{B}(\mathcal{H})$ such that

$$P_Q\Phi(u) = \Phi(u)P_Q = \Phi_Q(u), \quad u \in \mathcal{R}(X);$$
$$\Phi_Q \text{ is a representation of } \mathcal{R}(X) \text{ on } \mathcal{H}_Q \overset{\mathsf{def}}{=} P_Q\mathcal{H};$$
$$P_{Q_1}P_{Q_2} = P_{Q_2}P_{Q_1} = 0 \text{ if } Q_1 \text{ and } Q_2 \text{ are such with } Q_1Q_2 = Q_2Q_1 = 0;$$
$$\text{if } \|\Phi\| = 1 \text{ the projections } P_Q \text{ are orthogonal.}$$

Call Φ_Q the *Q–part of* Φ; this definition applies to the operator $T = \Phi(u_1)$ as well.

Referring to the above let us restate Theorem 5.3 of [**22**] as follows [4]

Theorem 5. *Let* Φ *be a representation of* $\mathcal{R}(X)$ *on* $\mathcal{H}$. *Suppose* $\boldsymbol{Q}$ *is a system of commuting projections having the property R and* $\{Q_\alpha\}_\alpha \subset \mathcal{B}(\boldsymbol{Q})$ *is composed of projections such that* $Q_\alpha Q_\beta = 0$ *for* $\alpha \neq \beta$ *then there exists* $S \in \boldsymbol{B}(\mathcal{H})$ *with* $S^{-1} \in \boldsymbol{B}(\mathcal{H})$ *such that*

$$S^{-1}\Phi(u)S = \bigoplus_\alpha \Phi_\alpha(u) \oplus \Phi_0(u), \quad u \in \mathcal{R}(X), \tag{5}$$

[4] These results have passed unnoticed because, presumably, the people recognized in the area have not found it deserving any attention; even in so authoritative monograph like [**16**] there is no mention of it though similarity is one of the leading topics therein; *winner takes all!*

where Φ_α is the Q_α–part of Φ and Φ_0 is the $\bigwedge_\alpha (I - Q_\alpha)$–part of Φ (the latter refers to the Boolean operation in $\mathcal{B}(\boldsymbol{Q})$).

When Φ is contractive, all the projections P_Q become contractive as well and there is no need to look for any similarity S; this was developed in [**15**] where a dilation free extension of results of [**18**] is treated, for some application to subnormal operators see [**14**]. There is one more instance when the same effect appears, see [**21**].

THEOREM 6. *Under the assumptions of* (Theorem 5) *the operator S appearing there can be chosen to be the identity operator provided there exists a system $\{\mu_{f,g}\}_{f,g}$ of elementary spectral measures of Φ such that*

$$\text{the measures } \mu_{f,f},\ f \in \mathcal{H}, \text{ are real;} \tag{6}$$

$$Q\mu_{f,f} \text{ are real too for any } Q \text{ in question.} \tag{7}$$

It is clear that the above happens because, due to (6) and (7), all the projections P_Q involved become selfadjoint. Notice also that, under these circumstances the result may be applicable to, the condition (7) is automatically satisfied.

Spectral sets and their representations. Given $K > 0$, call a compact set $X \subset \mathbb{C}$ a *K–spectral set* of $T \in \boldsymbol{B}(\mathcal{H})$

$$\|u(T)\| \leqslant K \sup\nolimits_{z \in X} |u(z)|, \quad \text{for all rational functions } u \text{ with poles off } X.$$

If K can be chosen to be 1 call X a *von Neumann spectral set* of T. Moreover, if for X and T there is K such that X is a K–spectral set of T, we say that X is just a *spectral set* of T [5].

Von Neumann's celebrated theorem stays that the unit disc is a von Neumann spectral set for a contraction. In fact, positive results in the matter compete with negative ones; mostly because the number one candidate as spectrum of an operator is, with all its possible oddities like holes, gives a real trouble. It seems that spectral sets (or one may prefer K-spectral sets) in fact involve two parameters X and K and this opens the doors to some activity, which has been done for long, including for those who like to optimize.

The Delyons result and its adherents. One of the results we have just had in mind is this [**8**] which follows. It is in a sense far going [6].

THEOREM 7. *Let T be an operator on a Hilbert space and X be a bounded convex subset of $\mathbb{C}$ containing $\mathfrak{W}(T)$. Then X is a spectral set of T.*

THEOREM 8. *Let X be as in* Theorem 7 *and assume that is has a piecewise $\mathcal{C}^1$ boundary ∂X. Denote by $\mathcal{C}(\partial X)$ the space of continuous functions on ∂X endowed with the uniform norm. Under the assumptions of* Theorem 7*, there exist a continuous linear operator S on $\mathcal{C}(\partial X)$ and a semispectral measure F on ∂X such that*

$$u(T) = \int_{\partial X} Su \, \mathrm{d}F$$

for any rational function u with poles off X.

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[5] Please notice a little deviation from the standard terminology.

[6] Except that in [**17**], cf.Theorem 9, we ought to mention also (some of) other which come up: [**6**] as well as [**7**], [**1**] and [**3**].

A few years later the above theorem was subsumed by Putinar and Sandberg [**17**] into the double layer potential theory of Carl Neumann, which resulted in a kind of normal dilation theorem. The latter can be read as follows.

THEOREM 9. *Let T be a bounded linear operator in a Hilbert space $\mathcal{H}$ and let X be a compact convex set which contains the numerical range of T. Then there exists a normal operator $N \in \boldsymbol{B}(\mathcal{K})$, acting on a larger space $\mathcal{K}$, with spectrum on the curve ∂X, such that:*

$$u(T) = 2P[(I+K)^{-1}u](N)P \tag{8}$$

for any function u continuous on X and harmonic in the interior of X. Here P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ and the linear continuous transformation $K\colon \mathcal{C}(\partial X) \to \mathcal{C}(\partial X)$ is the classical Neumann-Poincare singular integral operator.

Therefore, the relation between Theorem 8 and Theorem 9 is in

$$S = 2[(I+K)^{-1}].$$

We set up Theorem 9 into further inquiry ending in a little lemma. For any $f \in \mathcal{H}$ there is a unique measure μ_f such that

$$\int_{\partial X} u \,\mathrm{d}\mu_f = \langle 2P[(I+K)^{-1}u](N)Pf, f\rangle.$$

LEMMA 10. *The measures μ_f, $f \in \mathcal{H}$ are real.*

PROOF. Just a couple of words for the proof: as the Neumann-Poincaré integral operator of $\mathcal{C}(\partial X)$ into itself, which really K is, maps real functions into real themselves, the whole argument with Neumann series for $(I+K)^{-1}$ from the very bottom of p. 348 of [**17**] (after Theorem 1 therein) applied to a real u makes therefore the measures μ_f real. □

Remark 11. In [**21**] a generalization of ρ–dilatability was proposed, very much in flavour of (8). It was designed to generate real elementary measures like $\mathcal{C}_\rho$ operators do.

The orthogonal decomposition. Notice that either from Theorem 7 or from Theorem 9, depending on a kind of assumption on X on wants to impose, it follows that X is a spectral set of T. Consequently T generates a representation, say Φ, of $\mathcal{R}(X)$, according to (3).

Now we are in a position to state our decomposition result.

THEOREM 12. *Suppose $T \in \boldsymbol{B}(\mathcal{H})$ and $X \subset \mathbb{C}$ are as in* Theorem 9 *and there is a system of elementary spectral measures of T such that* (6) *holds. Suppose $\boldsymbol{Q}$ is a system of commuting projections having the property R, satisfying* (4) *and* (7) *and $\{Q_\alpha\}_\alpha \subset \mathcal{B}(\boldsymbol{Q})$ is composed of projections such that $Q_\alpha Q_\beta = 0$ for $\alpha \neq \beta$ then there exists a system $\{S_\alpha\}_\alpha \cup \{S_0\}$ of similarities in $\mathcal{H}_\alpha \overset{\mathrm{def}}{=} P_\alpha\mathcal{H}$ and $\mathcal{H}_0 \overset{\mathrm{def}}{=} P_0\mathcal{H}$ such that*

$$\Phi(u) = \bigoplus_\alpha S_\alpha^{-1}\Psi_\alpha(u)S_\alpha \oplus S_0^{-1}\Psi_0(u)S_0, \quad u \in \mathcal{R}(X), \tag{9}$$

where $S_\alpha^{-1}\Psi_\alpha S_\alpha$ is the Q_α–part of Φ and $S_0^{-1}\Phi_0 S_0$ is the $\bigwedge_\alpha(I-Q_\alpha)$–part of Φ. The representations Ψ_α and Ψ_0 are contractive.

Proof. The only thing which requires some explanation is appearance of the similarities S_α and S_0. The representations Ψ_α and Ψ_0 which can be get from (9) with $S = I$ according to Theorem 6, or rather the corresponding operators $T_\alpha = \Psi_\alpha(u_1)$ and $T_0 = \Psi_0(u_1)$ have their numerical ranges contained in X as well; this is due to (1). Theorem 2 of [**6**] tells us that each of those $T_\alpha = \Psi_\alpha(u_1)$'s as well as $T_0 = \Psi_0(u_1)$ are completely bounded. Now an application of Theorem 9.1, p. 120 of [**16**] generates the similarities in question. □

Notice that one of the new ingredients in Theorem 12 comparing to what is in [**21**] is the appearance of similarities within the orthogonal decomposition, or in other words, a kind of diagonalization with respect to the aforesaid orthogonal decomposition.

Particular cases. The two particular cases we can apply Theorem 12 to are the decompositions determined by those for sets of antisymmetry (Theorem 1) and Gleason parts (Theorem 3). To state the relevant results for an operator T which generates the representation Φ is a matter of necessity. Let us mention only that the assumptions of (4) and (7) to hold can be removed due to the nature of projections Q involved.

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